

# NORMAL FORMS OF REAL HYPERSURFACES WITH NONDEGENERATE LEVI FORM

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ABSTRACT. We present a proof of the existence and uniqueness theorem of a normalizing biholomorphic mapping to Chern-Moser normal form. The explicit form of the equation of a chain on a real hyperquadric is obtained. There exists a family of normal forms of real hypersurfaces including Chern-Moser normal form.

## 0. Introduction

Let  $M$  be an analytic real hypersurface with nondegenerate Levi form in a complex manifold and  $p$  be a point on  $M$ . Then it is known that there is a local coordinate system  $z^1, z^2, \dots, z^n, z^{n+1} \equiv w = u + iv$  with center at  $p$ , where  $M$  is locally defined by the equation

$$(0.1) \quad v = \langle z, z \rangle + \sum_{s,t \geq 2} F_{st}(z, \bar{z}, u),$$

where

- (1)  $\langle z, z \rangle \equiv z^1 \bar{z}^1 + \dots + z^e \bar{z}^e - z^{e+1} \bar{z}^{e+1} - \dots - z^n \bar{z}^n$  for a positive integer  $e$  in  $\frac{n}{2} \leq e \leq n$ ,
- (2)  $F_{st}(z, \bar{z}, u)$  is a real-analytic function of  $z, u$  for each pair  $(s, t) \in \mathbb{N}^2$ , which satisfies

$$F_{st}(\mu z, \nu \bar{z}, u) = \mu^s \nu^t F_{st}(z, \bar{z}, u),$$

for all complex numbers  $\mu, \nu$ ,

- (3) the functions  $F_{22}, F_{23}, F_{33}$  satisfy the following conditions:

$$\Delta F_{22} = \Delta^2 F_{23} = \Delta^3 F_{33} = 0,$$

where

$$\Delta \equiv D_1 \bar{D}_1 + \dots + D_e \bar{D}_e - D_{e+1} \bar{D}_{e+1} - \dots - D_n \bar{D}_n,$$

$$D_k = \frac{\partial}{\partial z^k}, \quad \bar{D}_k = \frac{\partial}{\partial \bar{z}^k}, \quad k = 1, \dots, n.$$

The local coordinate system (0.1) is called normal coordinate. The existence of a normal coordinate is a natural consequence of the following existence theorem of a normalizing biholomorphic mapping to Chern-Moser normal form.

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**Theorem 0.1** (Chern-Moser). *Let  $M$  be an analytic real hypersurface with nondegenerate Levi form at the origin in  $\mathbb{C}^{n+1}$  defined by the following equation:*

$$v = F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0.$$

*Then there is a biholomorphic mapping  $\phi$  such that*

$$(0.2) \quad \phi(M) : v = \langle z, z \rangle + \sum_{s,t \geq 2} F_{st}^*(z, \bar{z}, u)$$

*where*

$$(0.3) \quad \Delta F_{22}^* = \Delta^2 F_{23}^* = \Delta^3 F_{33}^* = 0.$$

We shall modify the proof given by Chern and Moser on the existence theorem(cf. [CM]) in order that the proof itself yields the uniqueness theorem as well:

**Theorem 0.2** (Chern-Moser). *Let  $M$  be a nondegenerate analytic real hypersurface in Theorem 0.1. Then the normalization  $\phi = (f, g)$  in  $\mathbb{C}^n \times \mathbb{C}$  is uniquely determined by the value*

$$\left. \frac{\partial f}{\partial z} \right|_0, \quad \left. \frac{\partial f}{\partial w} \right|_0, \quad \Re \left( \left. \frac{\partial g}{\partial w} \right|_0 \right), \quad \Re \left( \left. \frac{\partial^2 g}{\partial w^2} \right|_0 \right).$$

An analytic curve on a nondegenerate real hypersurface  $M$  is called a chain if it can be straightened by a normalization. We carefully examine the equation of a chain on  $M$  so that, in particular, we obtain the explicit form of the equation of a chain  $\gamma$  on a real hyperquadric:

$$\gamma : \begin{cases} z = p(\mu) \\ w = \mu + i\langle p(\mu), p(\mu) \rangle \end{cases}$$

where the function  $p(\mu)$  is a solution of the following ordinary differential equation(cf. [Pi])

$$p'' = \frac{2ip'\langle p', p' \rangle (1 + 3i\langle p, p' \rangle - i\langle p', p \rangle)}{(1 + i\langle p, p' \rangle - i\langle p', p \rangle) (1 + 2i\langle p, p' \rangle - 2i\langle p', p \rangle)}.$$

Normalizations of a real hypersurface  $M$  to Chern-Moser normal form is parameterized by a finite dimensional group  $H$  given by

$$\begin{pmatrix} \rho & 0 & 0 \\ -Ca & C & 0 \\ -r - i\langle a, a \rangle & 2ia^\dagger & 1 \end{pmatrix}$$

where

$$a^\dagger = (\overline{a^1}, \dots, \overline{a^e}, -\overline{a^{e+1}}, \dots, -\overline{a^n}).$$

The family of normalization of  $M$  shall depend analytically on the parameters

$$C = \left( \left. \frac{\partial f}{\partial z} \right|_0 \right), \quad -Ca = \left( \left. \frac{\partial f}{\partial w} \right|_0 \right), \quad \rho = \Re \left( \left. \frac{\partial g}{\partial w} \right|_0 \right), \quad 2\rho r = \Re \left( \left. \frac{\partial^2 g}{\partial w^2} \right|_0 \right).$$

We shall show that there is a family of normal forms such that

$$\begin{aligned} v &= \langle z, z \rangle + \sum_{s,t \geq 2} F_{st}(z, \bar{z}, u) && \text{for } \alpha = 0 \\ v &= -\frac{1}{2\alpha} \ln \{1 - 2\alpha \langle z, z \rangle\} + \sum_{s,t \geq 2} F_{st}(z, \bar{z}, u) && \text{for } \alpha \neq 0 \end{aligned}$$

where  $\alpha \in \mathbb{R}$  and

$$\begin{aligned}\Delta F_{22}(z, \bar{z}, u) &= \Delta^2 F_{23}(z, \bar{z}, u) = 0 \\ \Delta^3 F_{33}(z, \bar{z}, u) &= \beta \Delta^4 (F_{22}(z, \bar{z}, u))^2 \quad \text{for some } \beta \in \mathbb{R}.\end{aligned}$$

Normalization of a real hypersurface to any normal form among this family is parametrized by the group  $H$ .

### 1. EXISTENCE AND UNIQUENESS THEOREM

Let  $M$  be an analytic real hypersurface defined near the origin by

$$v = F(z, \bar{z}, u), \quad F|_0 = 0$$

and  $\Gamma : [-1, 1] \rightarrow M$  be an analytic real curve passing through the origin. Then the equation of  $\Gamma$  is given as follows:

$$\Gamma : \begin{cases} z = p(\mu), \\ w = q(\mu), \end{cases}$$

where  $p(0) = q(0) = 0$ . Since  $\Gamma$  is obviously tangent to  $M$  at the origin, we have the equality

$$q'(0) = \left(1 + i \frac{\partial F}{\partial u} \Big|_0\right) \Re q'(0) + i \left( \frac{\partial F}{\partial z^\alpha} \Big|_0 p^{\alpha'}(0) + \frac{\partial F}{\partial \bar{z}^\alpha} \Big|_0 \bar{p}^{\alpha'}(0) \right).$$

Then  $\Gamma$  is transversal to the complex tangent hyperplane of  $M$  at the origin if and only if the following condition holds

$$\Gamma_* \frac{d}{dt} \notin \ker \partial \left\{ \frac{w - \bar{w}}{2i} - F \left( z, \bar{z}, \frac{w + \bar{w}}{2} \right) \right\}.$$

Then the necessary and sufficient condition for the transversality is given by the inequality

$$\frac{\partial F}{\partial z^\alpha} \Big|_0 p^{\alpha'}(0) \neq \frac{1}{2i} \left(1 - i \frac{\partial F}{\partial u} \Big|_0\right) q'(0).$$

Thus, under the condition  $F_z|_0 = F_{\bar{z}}|_0 = 0$ , the transversality of  $\Gamma$  to the complex tangent hyperplane of  $M$  at the origin is equivalent to the inequality

$$\Re q'(0) = \left\{ 1 + \left( \frac{\partial F}{\partial u} \Big|_0 \right)^2 \right\}^{-1} q'(0) \neq 0.$$

Hence we suppose that  $M$  is an analytic real hypersurface defined by

$$(1.1) \quad v = F(z, \bar{z}, u),$$

where

$$F|_0 = F_z|_0 = F_{\bar{z}}|_0 = 0.$$

Let  $\Gamma$  be an analytic real curve on  $M$  transversal to the complex tangent hyperplane at the origin. Then the equation of  $\Gamma$  is uniquely given with a distinguished parameterization  $\mu$  as follows:

$$(1.2) \quad \Gamma : \begin{cases} z = p(\mu) \\ w = \mu + iF(p(\mu), \bar{p}(\mu), \mu) \end{cases}.$$

**Lemma 1.1.** *Let  $g(z, w)$  be a holomorphic function implicitly defined by the equations:*

$$(1.3) \quad \begin{aligned} g(z, w) - g(0, w) &= -2iF(p(w), \bar{p}(w), w) \\ &\quad + 2iF\left(z + p(w), \bar{p}(w), w + \frac{1}{2}\{g(z, w) - g(0, w)\}\right), \\ g(0, w) &= iF(p(w), \bar{p}(w), w). \end{aligned}$$

Let  $\phi$  be a biholomorphic mapping near the origin defined by

$$(1.4) \quad \begin{aligned} z &= z^* + p(w^*), \\ w &= w^* + g(z^*, w^*). \end{aligned}$$

Then the mapping  $\phi$  transforms the real hypersurface  $M$  such that  $M' \equiv \phi(M)$  is locally defined by an equation of the following form

$$v^* = \sum_{\min(s, t) \geq 1} F_{st}^*(z^*, \bar{z}^*, u^*)$$

and the curve  $\Gamma$  on  $M$  via the equation (1.2) is mapped on the  $u$ -curve,  $z = v = 0$ .

Note that the holomorphic function  $g(z, w)$  is well defined because of the condition

$$F|_0 = F_z|_0 = F_{\bar{z}}|_0 = 0,$$

which implies

$$g|_0 = \frac{\partial g}{\partial z}\bigg|_0 = \Re\left(\frac{\partial g}{\partial w}\bigg|_0\right) = 0.$$

Further, the mapping (1.4) is bijective at the origin. Hence the mapping (1.4) is biholomorphic near the origin for any analytic function  $p(u)$  such that  $p(0) = 0$ .

Suppose that the transformed real hypersurface  $M'$  is defined by

$$v^* = F^*(z^*, \bar{z}^*, u^*).$$

Then the mapping (1.4) yields the following equality:

$$(1.5) \quad F(z, \bar{z}, u) = F^*(z^*, \bar{z}^*, u^*) + \frac{1}{2i}\{g(z^*, u^* + iv^*) - \bar{g}(\bar{z}^*, u^* - iv^*)\},$$

where

$$\begin{aligned} z &= z^* + p(u^* + iv^*), \\ \bar{z} &= \bar{z}^* + \bar{p}(u^* - iv^*), \\ u &= u^* + \frac{1}{2}\{g(z^*, u^* + iv^*) + \bar{g}(\bar{z}^*, u^* - iv^*)\}. \end{aligned}$$

Since  $F$  and  $F^*$  are real analytic, we can consider  $z^*, \bar{z}^*, u^*$  as independent variables. Hence the condition of  $F^*(z^*, 0, u^*) = v^* = 0$  is equivalent via the equality (1.5) to the following equality:

$$(1.6) \quad g(z, u) - \overline{g(0, u)} = 2iF\left(z + p(u), \bar{p}(u), u + \frac{1}{2}\{g(z, u) + \overline{g(0, u)}\}\right).$$

Taking  $z = 0$  yields

$$(1.7) \quad g(0, u) - \overline{g(0, u)} = 2iF\left(p(u), \bar{p}(u), u + \frac{1}{2}\{g(0, u) + \overline{g(0, u)}\}\right).$$

Thus we easily see that

$$(1.8) \quad g(0, u) + \overline{g(0, u)} = 0$$

if and only if

$$g(0, u) = iF(p(u), \bar{p}(u), u).$$

Let  $\Gamma$  be a curve on  $M$  defined by a function  $p(u)$  via the equation (1.2). Then the mapping (1.4) maps the curve  $\Gamma$  onto the  $u$ -curve in  $\mathbb{C}^{n+1}$  if and only if the condition (1.8) on  $g(z, w)$  is satisfied.

By requiring the condition (1.8), the equality (1.6) reduces to

$$\begin{aligned} g(z, u) - g(0, u) = & -2iF(p(u), \bar{p}(u), u) \\ & + 2iF\left(z + p(u), \bar{p}(u), u + \frac{1}{2}\{g(z, u) - g(0, u)\}\right). \end{aligned}$$

Thus the equalities (1.6) and (1.7) are satisfied by the function  $g(z, w)$  defined in the equalities (1.3). This completes the proof of Lemma 1.1.

Note that the mapping (1.4) in Lemma 1.1 is completely determined by the analytic function  $p(u)$ . From the equality (1.3), we obtain the expansion of the holomorphic function  $g(z, w)$  as a power series of  $z$  up to order 3 inclusive as follows:

$$\begin{aligned} g(z, w) = & iF(p(w), \bar{p}(w), w) \\ & + 2i(1 - iF')^{-1}\{F_\alpha z^\alpha + F_{\alpha\beta} z^\alpha z^\beta + F_{\alpha\beta\gamma} z^\alpha z^\beta z^\gamma\} \\ & - 2(1 - iF')^{-2}\{F_\alpha z^\alpha F'_\beta z^\beta + F_\alpha z^\alpha F'_{\beta\gamma} z^\beta z^\gamma \\ & \quad + F_{\alpha\beta} z^\alpha z^\beta F'_\gamma z^\gamma\} \\ & - 2i(1 - iF')^{-3}\{F_\alpha z^\alpha (F'_\beta z^\beta)^2 + 2F_\alpha z^\alpha F_{\beta\gamma} z^\beta z^\gamma F'' \\ & \quad + (F_\alpha z^\alpha)^2 F'' + (F_\alpha z^\alpha)^2 F''_{\beta\gamma} z^\beta\} \\ & + 2(1 - iF')^{-4}\{3(F_\alpha z^\alpha)^2 F'_\beta z^\beta F'' + (F_\alpha z^\alpha)^3 \cdot F'''\} \\ & + 4i(1 - iF')^{-5}(F_\alpha z^\alpha)^3 (F'')^2 \\ & + O(z^4) \end{aligned} \tag{1.9}$$

where

$$\begin{aligned}
F_\alpha &= \sum_\alpha \left( \frac{\partial F}{\partial z^\alpha} \right) (p(w), \bar{p}(w), w), \\
F' &= \left( \frac{\partial F}{\partial u} \right) (p(w), \bar{p}(w), w), \\
F_{\alpha\beta} &= \frac{1}{2} \sum_{\alpha, \beta} \left( \frac{\partial^2 F}{\partial z^\alpha \partial z^\beta} \right) (p(w), \bar{p}(w), w), \\
F'_\alpha &= \sum_\alpha \left( \frac{\partial^2 F}{\partial z^\alpha \partial u} \right) (p(w), \bar{p}(w), w), \\
F'' &= \frac{1}{2} \left( \frac{\partial^2 F}{\partial u^2} \right) (p(w), \bar{p}(w), w), \\
F_{\alpha\beta\gamma} &= \frac{1}{6} \sum_{\alpha, \beta, \gamma} \left( \frac{\partial^3 F}{\partial z^\alpha \partial z^\beta \partial z^\gamma} \right) (p(w), \bar{p}(w), w), \\
F'_{\alpha\beta} &= \frac{1}{2} \sum_{\alpha, \beta} \left( \frac{\partial^3 F}{\partial z^\alpha \partial z^\beta \partial u} \right) (p(w), \bar{p}(w), w), \\
F''_\alpha &= \frac{1}{2} \sum_\alpha \left( \frac{\partial^3 F}{\partial z^\alpha \partial u^2} \right) (p(w), \bar{p}(w), w), \\
F''' &= \frac{1}{6} \left( \frac{\partial^3 F}{\partial u^3} \right) (p(w), \bar{p}(w), w).
\end{aligned}$$

By Lemma 1.1, we have the following condition on the real hypersurface  $M'$ :

$$v = O(z\bar{z}).$$

Thus it suffices to obtain terms up to  $v^2$  inclusive in order that we compute the functions

$$F_{st}^*(z, \bar{z}, u)$$

of  $M'$  up to the type  $(s, t)$ ,  $s + t \leq 5$  inclusive.

We obtain the expansions of  $p^\alpha(u + iv)$  and  $p^{\bar{\beta}}(u + iv)$  as power series of  $v$  as follows:

$$\begin{aligned}
p^\alpha(u + iv) &= p^\alpha + p^{\alpha'} \cdot iv + p^{\alpha''} \cdot \frac{(iv)^2}{2} + O(v^3) \\
p^{\bar{\beta}}(u + iv) &= p^{\bar{\beta}} + p^{\bar{\beta}'} \cdot iv + p^{\bar{\beta}''} \cdot \frac{(iv)^2}{2} + O(v^3).
\end{aligned}$$

By using this expansion, we expand the holomorphic function  $g(z, w)$  as a power series of  $z$  and  $v$  in (1.9) as follows:

$$\begin{aligned}
g(z, w) &= \sum_{k,l=0}^{\infty} g_k^{(l)}(z, u) \frac{(iv)^l}{l!} \\
&= iF(p(u), \bar{p}(u), u) + g'_0(0, u)iv + g''_0(0, u) \frac{(iv)^2}{2} \\
&\quad + g_1(z, u) + g'_1(z, u)iv + g''_1(z, u) \frac{(iv)^2}{2} \\
&\quad + g_2(z, u) + g'_2(z, u)iv + g_3(z, u) \\
&\quad + O(z^4) + O(z^3v) + O(z^2v^2) + O(zv^3) + O(v^3)
\end{aligned}$$

where

$$g_k^{(l)}(\mu z, u) = \mu^k g_k^{(l)}(z, u)$$

for all complex number  $\mu$ .

We easily see that the function  $g_k^{(l)}(z, u)$  depends analytically on the functions  $p(u)$  and  $\bar{p}(u)$ , polynomially on the derivatives of  $p(u)$  and  $\bar{p}(u)$  up to order  $l$  inclusive such that the order sum of the derivatives in each term of  $g_k^{(l)}(z, u)$  is less than or equal to the integer  $l$ . In low order terms, we obtain

$$\begin{aligned}
g_0(0, u) &= iF(p(u), \bar{p}(u), u) = O(u) \\
g'_0(0, u) &= iF_\alpha p^{\alpha'} + iF_{\bar{\beta}} \bar{p}^{\bar{\beta}'} + iF' = O(1) \\
g''_0(0, u) &= iF_\alpha p^{\alpha''} + iF_{\bar{\beta}} \bar{p}^{\bar{\beta}''} + 2iF_{\alpha\beta} p^{\alpha'} \bar{p}^{\beta'} + 2iF_{\alpha\bar{\beta}} p^{\alpha'} \bar{p}^{\bar{\beta}'} + 2iF_{\bar{\alpha}\beta} \bar{p}^{\alpha'} p^{\beta'} \\
&\quad + 2iF'_{\alpha} p^{\alpha'} + 2iF_{\bar{\beta}}^L \bar{p}^{\bar{\beta}'} + 2iF''
\end{aligned}$$

and

$$\begin{aligned}
g_1(z, u) &= 2i(1 - F')^{-1} \cdot F_\alpha z^\alpha = O(zu) \\
g'_1(z, u) &= 2i(1 - F')^{-1} \left\{ 2F_{\alpha\beta} z^\alpha p^{\beta'} + F_{\alpha\bar{\beta}} z^\alpha \bar{p}^{\bar{\beta}'} + F'_\alpha z^\alpha \right\} \\
&\quad - 2(1 - F')^{-1} \left\{ F'_\alpha p^{\alpha'} + F_{\bar{\beta}}^L \bar{p}^{\bar{\beta}'} + 2F'' \right\} F_\alpha z^\alpha \\
&= 4i F_{\alpha\beta}|_0 z^\alpha p^{\beta'} + 2i F_{\alpha\bar{\beta}}|_0 z^\alpha \bar{p}^{\bar{\beta}'} + O(zu) \\
g_2(z, u) &= 2i(1 - F')^{-1} F_{\alpha\beta} z^\alpha z^{\beta'} - 2(1 - F')^{-2} F_\alpha z^\alpha F'_{\beta} z^{\beta'} \\
&\quad - 2i(1 - F')^{-3} (F_\alpha z^\alpha)^2 F'' \\
&= 2i F_{\alpha\beta}|_0 z^\alpha z^{\beta'} + O(z^2u)
\end{aligned}$$

The real hypersurface  $M'$  is defined by the following equation:

$$\begin{aligned}
v = F &\left( z + p(u + iv), \bar{z} + \bar{p}(u - iv), u + \frac{1}{2} \{g(z, u + iv) + \bar{g}(\bar{z}, u - iv)\} \right) \\
(1.10) \quad &- \frac{1}{2i} \{g(z, u + iv) - \bar{g}(\bar{z}, u - iv)\}.
\end{aligned}$$

We expand the right hand side of the equation (1.10) in low order terms of  $v$  as follows:

$$\begin{aligned} & F\left(z + p(u + iv), \bar{z} + \bar{p}(u - iv), u + \frac{1}{2}\{g(z, u + iv) + \bar{g}(\bar{z}, u - iv)\}\right) \\ & \quad - \frac{1}{2i}\{g(z, u + iv) - \bar{g}(\bar{z}, u - iv)\} \\ & = A(z, \bar{z}, u) + vB(z, \bar{z}, u) + v^2C(z, \bar{z}, u) + O(v^3). \end{aligned}$$

Then we obtain

$$\begin{aligned} A(z, \bar{z}, u) &= F(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) - \Im g(z, u) \\ B(z, \bar{z}, u) &= iF_\alpha(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))p^{\alpha'}(u) \\ &\quad - iF_{\bar{\beta}}(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))p^{\bar{\beta}'}(u) \\ &\quad - F'(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))\Im g'(z, u) \\ &\quad - \Re g'(z, u) \\ C(z, \bar{z}, u) &= -F_{\alpha\beta}(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))p^{\alpha'}(u)p^{\beta'}(u) \\ &\quad + F_{\alpha\bar{\beta}}(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))p^{\alpha'}(u)p^{\bar{\beta}'}(u) \\ &\quad - F_{\bar{\alpha}\beta}(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))p^{\bar{\alpha}'}(u)p^{\beta'}(u) \\ &\quad - iF'_\alpha(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))p^{\alpha'}(u)\Im g'(z, u) \\ &\quad + iF'_{\bar{\beta}}(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))p^{\bar{\beta}'}(u)\Im g'(z, u) \\ &\quad + F''(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))(\Im g'(z, u))^2 \\ &\quad - \frac{1}{2}F_\alpha(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))p^{\alpha''}(u) \\ &\quad - \frac{1}{2}F_{\bar{\beta}}(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))p^{\bar{\beta}''}(u) \\ &\quad - \frac{1}{2}F'(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u))\Re g''(z, u) \\ &\quad + \frac{1}{2}\Im g''(z, u) \end{aligned}$$

where

$$\begin{aligned} g'(z, u) &= \left(\frac{\partial g}{\partial w}\right)(z, u) \\ g''(z, u) &= \left(\frac{\partial^2 g}{\partial w^2}\right)(z, u). \end{aligned}$$

We decompose the functions  $A(z, \bar{z}, u)$ ,  $B(z, \bar{z}, u)$ ,  $C(z, \bar{z}, u)$  as follows:

$$\begin{aligned} A(z, \bar{z}, u) &= \sum_{\min(s, t) \geq 1} A_{st}(z, \bar{z}, u), \\ B(z, \bar{z}, u) &= \sum_{\min(s, t) \geq 0} B_{st}(z, \bar{z}, u), \\ C(z, \bar{z}, u) &= \sum_{\min(s, t) \geq 0} C_{st}(z, \bar{z}, u). \end{aligned}$$

We easily see the following facts:



- (1)  $A_{st}(z, \bar{z}, u)$  depends analytically on the functions  $p(u)$  and  $\bar{p}(u)$ .
- (2)  $B_{st}(z, \bar{z}, u)$  depends analytically on the function  $p(u)$  and  $\bar{p}(u)$ , at most linearly on the derivative  $p'(u)$  and  $\bar{p}'(u)$ .
- (3)  $C_{st}(z, \bar{z}, u)$  depends analytically on the function  $p(u)$  and  $\bar{p}(u)$ , at most quadratically on the derivative  $p'(u)$  and  $\bar{p}'(u)$ , at most linearly on the derivative  $p''(u)$  and  $\bar{p}''(u)$  such that the derivative order sum of the derivatives of  $p(u)$  and  $\bar{p}(u)$  in each term is less than or equal to 2.

**Lemma 1.2.** *The functions  $C_{0t}(z, \bar{z}, u)$ ,  $t \in \mathbb{N}$ , do not depend on the derivative  $p''(u)$  and  $\bar{p}''(u)$ .*

This claim is easily verified by observing that the following functions

$$C(z, \bar{z}, u) \quad \text{and} \quad -\frac{1}{2} \left( \frac{\partial^2 A}{\partial u^2} \right) (z, \bar{z}, u)$$

depend in the same manner on the derivative  $p''(u)$  and  $\bar{p}''(u)$ , because

$$A(z, \bar{z}, u + iv) = A(z, \bar{z}, u) + iv \left( \frac{\partial A}{\partial u} \right) (z, \bar{z}, u) - \frac{v^2}{2} \left( \frac{\partial^2 A}{\partial u^2} \right) (z, \bar{z}, u) + \cdots$$

By the way,  $A(z, 0, u) = 0$  is the defining equation of the function  $g(z, u)$  in Lemma 1.1 with  $g(0, u) = iF(p(u), \bar{p}(u), u)$ . Thus we have the following identities

$$\left( \frac{\partial A}{\partial u} \right) (z, 0, u) = \left( \frac{\partial^2 A}{\partial u^2} \right) (z, 0, u) = \cdots = 0.$$

Note that the identity

$$\left( \frac{\partial^2 A}{\partial u^2} \right) (z, 0, u) = 0$$

gives the desired relation between the terms having  $p''$  and  $\bar{p}''$  and the terms not having  $p''$  and  $\bar{p}''$  so that we verify the function  $C(z, 0, u)$  is independent of the derivative  $p''(u)$  and  $\bar{p}''(u)$ . This proves the claim in Lemma 1.2.

Explicitly, we compute the expansion of the right hand side of the equation (1.10) in low order terms so that

$$\begin{aligned} v = & A_{11}(z, \bar{z}, u) + A_{22}(z, \bar{z}, u) + A_{12}(z, \bar{z}, u) + A_{13}(z, \bar{z}, u) + A_{23}(z, \bar{z}, u) \\ & + A_{21}(z, \bar{z}, u) + A_{31}(z, \bar{z}, u) + A_{32}(z, \bar{z}, u) \} \\ & + v \{ B_{00}(z, \bar{z}, u) + B_{11}(z, \bar{z}, u) + B_{01}(z, \bar{z}, u) + B_{02}(z, \bar{z}, u) + B_{12}(z, \bar{z}, u) \\ & + B_{10}(z, \bar{z}, u) + B_{20}(z, \bar{z}, u) + B_{21}(z, \bar{z}, u) \} \\ & + v^2 \{ C_{00}(z, \bar{z}, u) + C_{01}(z, \bar{z}, u) + C_{10}(z, \bar{z}, u) \} \\ & + O(z^1 \bar{z}^4) + O(z^4 \bar{z}^1) + O(vz^3) + O(v\bar{z}^3) + O(v^3) \\ & + \sum_{\min(s,t) \geq 1, s+t \geq 6} O(z^s \bar{z}^t) + \sum_{s+t \geq 4} O(vz^s \bar{z}^t) + \sum_{s+t \geq 2} O(v^2 z^s \bar{z}^t), \end{aligned}$$

where

$$\begin{aligned}
A_{11}(z, \bar{z}, u) &= F_{\alpha\bar{\beta}} z^\alpha \bar{z}^\beta - i(1 + iF')^{-1} F'_\alpha z^\alpha F_{\bar{\beta}} \bar{z}^\beta \\
&\quad + i(1 - iF')^{-1} F_\alpha z^\alpha F'_{\bar{\beta}} \bar{z}^\beta \\
&\quad + 2(1 + iF')^{-1} (1 - iF')^{-1} F'' F_\alpha z^\alpha F_{\bar{\beta}} \bar{z}^\beta \\
&= \langle z, z \rangle + O(z\bar{z}u) \\
B_{00}(z, \bar{z}, u) &= i(1 + iF')^{-1} F_\alpha p^{\alpha'} - i(1 - iF')^{-1} F_{\bar{\beta}} \bar{p}^{\bar{\beta}'} - (F')^2 \\
&= O(1) \\
B_{01}(z, \bar{z}, u) &= 2iF_{\alpha\bar{\beta}} p^{\alpha'} \bar{z}^\beta + 2F'' F_{\bar{\alpha}} z^\alpha + i(1 + iF') F_{\bar{\alpha}} z^\alpha \\
&\quad + iF'_\alpha p^{\alpha'} (1 + iF')^{-1} F_{\bar{\alpha}} z^\alpha \\
&\quad + 2i \left( F_\alpha p^{\alpha'} + F'_{\bar{\beta}} \bar{p}^{\bar{\beta}'} \right) F'' (1 + iF')^{-1} F_{\bar{\gamma}} \bar{z}^\gamma
\end{aligned}$$

Then we obtain

$$\begin{aligned}
v &= F_{11}^*(z, \bar{z}, u) + F_{22}^*(z, \bar{z}, u) + F_{12}^*(z, \bar{z}, u) + F_{13}^*(z, \bar{z}, u) + F_{23}^*(z, \bar{z}, u) \\
&\quad + F_{21}^*(z, \bar{z}, u) + F_{31}^*(z, \bar{z}, u) + F_{32}^*(z, \bar{z}, u) \\
&\quad + O(z\bar{z}^4) + O(z^4\bar{z}) + \sum_{\min(s,t) \geq 1, s+t \geq 6} O(z^s \bar{z}^t),
\end{aligned}$$

where

$$\begin{aligned}
F_{11}^* &= (1 - B_{00})^{-1} A_{11} \\
F_{12}^* &= (1 - B_{00})^{-1} A_{12} + (1 - B_{00})^{-2} A_{11} B_{01} \\
F_{13}^* &= (1 - B_{00})^{-1} A_{13} + (1 - B_{00})^{-2} (A_{11} B_{02} + A_{12} B_{01}) \\
&\quad + (1 - B_{00})^{-3} A_{11} B_{01}^2 \\
F_{22}^* &= (1 - B_{00})^{-1} A_{22} + (1 - B_{00})^{-2} (A_{11} B_{11} + A_{12} B_{10} + A_{21} B_{01}) \\
&\quad + (1 - B_{00})^{-3} (2A_{11} B_{01} B_{10} + A_{11}^2 C_{00}) \\
F_{23}^* &= (1 - B_{00})^{-1} A_{23} + (1 - B_{00})^{-2} (A_{11} B_{12} + A_{12} B_{11} + A_{21} B_{02} \\
&\quad + A_{13} B_{10} + A_{22} B_{01}) \\
&\quad + (1 - B_{00})^{-3} (2A_{11} B_{01} B_{11} + 2A_{11} B_{10} B_{02} + 2A_{12} B_{01} B_{10} \\
&\quad + A_{21} B_{01}^2 + A_{11}^2 C_{01} + 2A_{11} A_{12} C_{00}) \\
&\quad + 3(1 - B_{00})^{-4} (A_{11} B_{01}^2 B_{10} + A_{11}^2 B_{01} C_{00}).
\end{aligned}$$

By Lemma 1.2, the functions  $F_{22}^*, F_{23}^*$  does not depend on the derivative  $p''$  and  $\bar{p}''$ , and the dependence of the coefficients in  $F_{22}^*, F_{23}^*$  on the derivative  $p'$  and  $\bar{p}'$  is of the form:

$$(1.11) \quad \frac{A_1(u, p, \bar{p}, p', \bar{p}')}{(1 - B_{00})^3},$$

and

$$(1.12) \quad \frac{A_2(u, p, \bar{p}, p', \bar{p}')}{(1 - B_{00})^4},$$

where  $A_1$  depends analytically on  $u, p, \bar{p}$  and at most quadratically on  $p', \bar{p}'$  and  $A_2$  depends analytically on  $u, p, \bar{p}$ , at most cubically on  $p', \bar{p}'$ .

For future reference, we analyze the terms containing the first order derivatives  $p', \bar{p}'$  in  $B_{00}$  and  $B_{01}$  so that

$$\begin{aligned} O(up') + O(u\bar{p}') & \quad \text{in } B_{00}(z, \bar{z}, u) \\ 2i\langle p', z \rangle + O(\bar{z}up') + O(\bar{z}u\bar{p}') & \quad \text{in } B_{01}(z, \bar{z}, u). \end{aligned}$$

Thus analyzing the terms containing the second order derivatives  $p''$  and  $\bar{p}''$  in

$$\left( \frac{\partial F_{11}^*}{\partial u} \right) (z, \bar{z}, u) \quad \text{and} \quad \left( \frac{\partial F_{12}^*}{\partial u} \right) (z, \bar{z}, u),$$

we obtain

$$\begin{aligned} O(z\bar{z}up'') + O(z\bar{z}u\bar{p}'') & \quad \text{in } \left( \frac{\partial F_{11}^*}{\partial u} \right) (z, \bar{z}, u) \\ (1.13) \quad 2i\langle z, z \rangle \langle p'', z \rangle + O(z\bar{z}^2up'') + O(z\bar{z}^2u\bar{p}'') & \quad \text{in } \left( \frac{\partial F_{12}^*}{\partial u} \right) (z, \bar{z}, u). \end{aligned}$$

Then we are ready to present a proof of the existence theorem for Chern-Moser normal form.

**Theorem 1.3** (Chern-Moser). *There is a biholomorphic mapping  $\phi$  which transforms  $M$  to a real hypersurface of the following form:*

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u),$$

where

$$(1.14) \quad \Delta^2 F_{23} = 0.$$

Geometrically, there exists a unique analytic curve  $\Gamma$  on  $M$  which passes through the origin and is tangent to a vector transversal to the complex tangent hyperplane at the origin and which is mapped onto the  $u$ -curve by the biholomorphic mapping  $\phi$ . Further, there exists a biholomorphic mapping  $\phi_1$  which, in addition to (1.14), achieves the following conditions:

$$\Delta F_{22} = \Delta^3 F_{33} = 0.$$

Let  $M'$  be a real hypersurface obtained in Lemma 1.1 by the biholomorphic mapping (1.4), which is defined by the following equation:

$$v = \sum_{\min(s,t) \geq 1} F_{st}^*(z, \bar{z}, u).$$

Then there is a unique analytic function  $D(z, u)$  (cf. [CM]) such that

$$F_{11}^*(z + D(z, u), \bar{z}, u) = \sum_{s \geq 1} F_{s1}^*(z, \bar{z}, u),$$

and the function  $D(z, u)$  satisfies the condition

$$D(0, u) = D_z(0, u) = 0.$$

Thus  $D(z, u)$  depends analytically of  $u, p, \bar{p}$  and rationally of the derivative  $p', \bar{p}'$ .

We decompose the function  $D(z, u)$  such that

$$D(z, u) = \sum_{s \geq 2} D_s(z, u),$$

where

$$D_s(\mu z, u) = \mu^s D_s(z, u) \quad \text{for all } \mu \in \mathbb{C}.$$

Then the functions  $D_2(z, u), D_3(z, u)$  are given by

$$\begin{aligned} A_{11}(D_2(z, u), \bar{z}, u) &= A_{21} + (1 - B_{00})^{-1} A_{11} B_{10} \\ A_{11}(D_3(z, u), \bar{z}, u) &= A_{31} + (1 - B_{00})^{-1} (A_{11} B_{20} + A_{21} B_{10}) \\ &\quad + (1 - B_{00})^{-2} A_{11} B_{10}^2. \end{aligned}$$

Note that  $D_2(z, u), D_3(z, u)$  do not depend of the second order derivative  $p'', \bar{p}''$ .

Then we obtain

$$\begin{aligned} v &= \sum_{\min(s,t) \geq 1} F_{st}^*(z, \bar{z}, u) \\ &= F_{11}^*(z, \bar{z}, u) + F_{11}^*\left(z, \overline{D(z, u)}, u\right) + F_{11}^*(D(z, u), \bar{z}, u) \\ &\quad + \sum_{\min(s,t) \geq 2} F_{st}^*(z, \bar{z}, u) \\ &= F_{11}^*\left(z + D(z, u), \overline{z + D(z, u)}, u\right) + \sum_{\min(s,t) \geq 2} G_{st}(z, \bar{z}, u). \end{aligned}$$

We notice

$$\begin{aligned} G_{22}(z, \bar{z}, u) &= F_{22}^*(z, \bar{z}, u) - F_{11}^*\left(D_2(z, u), \overline{D_2(z, u)}, u\right) \\ G_{23}(z, \bar{z}, u) &= F_{23}^*(z, \bar{z}, u) - F_{11}^*\left(D_2(z, u), \overline{D_3(z, u)}, u\right). \end{aligned}$$

We easily see that the functions  $G_{22}, G_{23}$  depend on  $u, p, \bar{p}, p', \bar{p}'$  in the same form as respectively in (1.11) and (1.12).

We take an analytic function  $E(u)$  such that

$$F_{11}^*(z, \bar{z}, u) = \langle E(u)z, E(u)z \rangle, \quad \text{and} \quad E(0) = id_{n \times n}.$$

Note that the function  $E(u)$  is determined up to the following relation:

$$E_1(u) = U(u)E(u),$$

where

$$(1.15) \quad \langle U(u)z, U(u)z \rangle = \langle z, z \rangle, \quad \text{and} \quad U(0) = id_{n \times n}.$$

Then the biholomorphic mapping defined by the following equation:

$$\begin{aligned} z^* &= E(w)\{z + D(z, w)\}, \\ w^* &= w, \end{aligned}$$

transforms  $M'$  to a real hypersurface of the following form:

$$(1.16) \quad v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} H_{st}(z, \bar{z}, u).$$

By the way, we still obtain a real hypersurface in (1.16) by a biholomorphic mapping as follows:

$$(1.17) \quad \begin{aligned} z^* &= U(w)E(w)\{z + D(z, w)\}, \\ w^* &= w, \end{aligned}$$

where the holomorphic function  $U(w)$  satisfy the condition (1.15).

By using the expansion

$$\begin{aligned} E(u) &= E(w) - ivE'(w) + \cdots \\ U(u) &= U(w) - ivU'(w) + \cdots, \end{aligned}$$

we obtain

$$\begin{aligned} v &= F_{11}^* \left( z + D(z, u), \overline{z + D(z, u)}, u \right) + \sum_{\min(s, t) \geq 2} G_{st}(z, \bar{z}, u) \\ &= \langle E(u)(z + D(z, u)), E(u)(z + D(z, u)) \rangle + \sum_{\min(s, t) \geq 2} G_{st}(z, \bar{z}, u) \\ &= \langle U(u)E(u)(z + D(z, u)), U(u)E(u)(z + D(z, u)) \rangle + \sum_{\min(s, t) \geq 2} G_{st}(z, \bar{z}, u) \\ &= \langle U(w)E(w)(z + D(z, w)), U(w)E(w)(z + D(z, w)) \rangle \\ &\quad - iv \langle U'(w)E(w)(z + D(z, w)), U(w)E(w)(z + D(z, w)) \rangle \\ &\quad + iv \langle U(w)E(w)(z + D(z, w)), U'(w)E(w)(z + D(z, w)) \rangle \\ &\quad - iv \langle \{E'(w)(z + D(z, w)) + E(w)D_u(z, w)\}, E(w)(z + D(z, w)) \rangle \\ &\quad + iv \langle E(w)(z + D(z, w)), \{E'(w)(z + D(z, w)) + E(w)D_u(z, w)\} \rangle \\ (1.18) \quad &+ O(z\bar{z}v^2) + \sum_{\min(s, t) \geq 2} G_{st}(z, \bar{z}, u), \end{aligned}$$

where

$$\begin{aligned} w &= u + iv, \\ U'(u) &= \frac{dU}{du}(u), \quad E'(u) = \frac{dE}{du}(u), \\ D_u(z, w) &= \left( \frac{\partial D}{\partial u} \right) (z, w). \end{aligned}$$

By introducing a holomorphic variable  $z^{\natural} = z + D(z, w)$ , we obtain from the equation (1.18):

$$\begin{aligned} v &= \langle U(w)E(w)z^{\natural}, U(w)E(w)z^{\natural} \rangle + G_{22}(z^{\natural}, \bar{z}^{\natural}, u) \\ &\quad - i \langle E(u)z^{\natural}, E(u)z^{\natural} \rangle \{ \langle U'(u)E(u)z^{\natural}, U(u)E(u)z^{\natural} \rangle \\ &\quad - \langle U(u)E(u)z^{\natural}, U'(u)E(u)z^{\natural} \rangle \\ &\quad - i \langle E(u)z^{\natural}, E(u)z^{\natural} \rangle \{ \langle E'(u)z^{\natural}, E(u)z^{\natural} \rangle - \langle E(u)z^{\natural}, E'(u)z^{\natural} \rangle \} \\ &\quad + G_{23}^*(z^{\natural}, \bar{z}^{\natural}, u) + G_{32}^*(z^{\natural}, \bar{z}^{\natural}, u). \\ (1.19) \quad &+ \sum_{\min(s, t) \geq 2, s+t \geq 6} G_{st}^*(z^{\natural}, \bar{z}^{\natural}, u). \end{aligned}$$

where

$$\begin{aligned}
(1.20) \quad G_{23}^*(z, \bar{z}, u) &= G_{23}(z, \bar{z}, u) + iF_{11}^*(z, \bar{z}, u)F_{11}^* \left( z, \overline{\left( \frac{\partial D_2}{\partial u} \right)}(z, u), u \right) \\
&\quad - \sum_{\beta} \left( \frac{\partial G_{22}}{\partial \bar{z}^{\beta}} \right) (z, \bar{z}, u) \overline{D_2^{\beta}(z, u)} \\
&= G_{23}(z, \bar{z}, u) + iF_{11}^*(z, \bar{z}, u) \left( \frac{\partial F_{12}^*}{\partial u} \right) (z, \bar{z}, u) \\
&\quad - iF_{11}^*(z, \bar{z}, u) \left( \frac{\partial F_{11}^*}{\partial u} \right) (z, \overline{D_2(z, u)}, u) \\
&\quad - \sum_{\beta} \left( \frac{\partial G_{22}}{\partial \bar{z}^{\beta}} \right) (z, \bar{z}, u) \overline{D_2^{\beta}(z, u)}
\end{aligned}$$

By the equalities (1.13), the dependence of the functions  $G_{23}^*$  on the second order derivatives  $p''$  and  $\bar{p}''$  is given as follows

$$-2\langle z, z \rangle^2 \langle p'', z \rangle + O(z^2 \bar{z}^3 u p'') + O(z^2 \bar{z}^3 u \bar{p}'') \quad \text{in } G_{23}^*(z, \bar{z}, u).$$

Notice that the function  $G_{23}^*(z, \bar{z}, u)$  in (1.19) and (1.20) is independent of the function  $U(u)$ .

Therefore after the biholomorphic mapping in (1.17), we obtain

$$\begin{aligned}
v &= \langle z^*, z^* \rangle + H_{22}(z^*, \bar{z}^*, u) + H_{23}(z^*, \bar{z}^*, u) + H_{32}(z^*, \bar{z}^*, u) \\
&\quad + \sum_{\min(s,t) \geq 2, s+t \geq 6} H_{st}(z^*, \bar{z}^*, u).
\end{aligned}$$

where

$$\begin{aligned}
H_{22}(z, \bar{z}, u) &= G_{22} \left( E(u)^{-1} U(u)^{-1} z, \overline{E(u)^{-1} U(u)^{-1} z}, u \right) \\
&\quad - i \langle z, z \rangle \{ \langle U'(u) U(u)^{-1} z, z \rangle - \langle z, U'(u) U(u)^{-1} z \rangle \} \\
&\quad - i \langle z, z \rangle \{ \langle E'(u) E(u)^{-1} U(u)^{-1} z, U(u)^{-1} z \rangle \\
&\quad \quad - \langle U(u)^{-1} z, E'(u) E(u)^{-1} U(u)^{-1} z \rangle \}
\end{aligned}$$

and the dependence of  $H_{23}(z, \bar{z}, u)$  on  $p'', \bar{p}''$  is as follows:

$$H_{23}(z, \bar{z}, 0) = -2\langle z, z \rangle^2 \langle p''(0), z \rangle + K_{23}(z, \bar{z}, 0; p'(0), \overline{p'(0)}).$$

By using the following identity

$$\Delta^2 \{ \langle z, z \rangle^2 \langle p, z \rangle \} = 2(n+1)(n+2) \langle p, z \rangle,$$

the equation  $\Delta^2 H_{23} = 0$  is a second order ordinary differential equation

$$A_1 p'' + A_2 \bar{p}'' = B$$

where

- (1)  $A_1, A_2$  are  $n \times n$  matrix valued functions and  $B$  is  $\mathbb{C}^n$ -valued function,
- (2)  $A_1 = id_{n \times n} + O(u)$  and  $A_2 = O(u)$ ,
- (3)  $A_1, A_2, B$  depend analytically of  $u, p, \bar{p}$ ,
- (4)  $A_1, A_2$  depend at most linearly of  $p', \bar{p}'$ ,
- (5)  $B$  depends at most cubically of  $p', \bar{p}'$ .

Then we obtain

$$(1.21) \quad \begin{aligned} p'' &= Q(u, p, \bar{p}, p', \bar{p}') \\ &\equiv \left( A_1 - A_2 \overline{A_1^{-1} A_2} \right)^{-1} \left( B - A_2 \overline{A_1^{-1} B} \right) \end{aligned}$$

where the function  $Q$  depends rationally on the derivatives  $p', \bar{p}'$ .

Therefore there exists a unique analytic curve  $\Gamma$  on  $M$  which passes through the origin and is tangent to a vector transversal to the complex tangent hyperplane at the origin and which is mapped by a biholomorphic mapping into the  $u$ -curve.

Since  $\langle U(u)z, U(u)z \rangle = \langle z, z \rangle$ , we have identities

$$\begin{aligned} \langle U'(u)U(u)^{-1}z, z \rangle + \langle z, U'(u)U(u)^{-1}z \rangle &= 0 \\ \text{Tr}(U'(u)U(u)^{-1}) + \overline{\text{Tr}(U'(u)U(u)^{-1})} &= 0, \end{aligned}$$

where

$$\text{Tr}(A) = \text{trace of } z \rightarrow Az,$$

Then the equation  $\Delta H_{22} = 0$  is given as follows:

$$(1.22) \quad \begin{aligned} &\langle U(u)^{-1}U'(u)z, z \rangle + \frac{1}{2(n+2)} \langle z, z \rangle \text{Tr}(U(u)^{-1}U'(u)) \\ &= \frac{1}{2i(n+2)} \Delta G_{22} \left( E(u)^{-1}z, \overline{E(u)^{-1}z}, u \right) \\ &\quad - \frac{1}{2} \{ \langle E'(u)E(u)^{-1}z, z \rangle - \langle z, E'(u)E(u)^{-1}z \rangle \} \\ &\quad - \frac{1}{2(n+2)} \langle z, z \rangle \left\{ \text{Tr}(E'(u)E(u)^{-1}) - \overline{\text{Tr}(E'(u)E(u)^{-1})} \right\}. \end{aligned}$$

By using the following identities

$$\begin{aligned} \Delta \{ \langle z, z \rangle \langle Az, z \rangle \} &= (n+2) \langle Az, z \rangle + \text{Tr}(A) \langle z, z \rangle \\ \Delta^2 \{ \langle z, z \rangle \langle Az, z \rangle \} &= 2(n+1) \text{Tr}(A), \end{aligned}$$

we obtain

$$\begin{aligned} \langle z, z \rangle \text{Tr}(U(u)^{-1}U'(u)) &= \frac{1}{4i(n+1)} \Delta^2 G_{22} \left( E(u)^{-1}z, \overline{E(u)^{-1}z}, u \right) \\ &\quad - \frac{1}{2} \left\{ \text{Tr}(E'(u)E(u)^{-1}) - \overline{\text{Tr}(E'(u)E(u)^{-1})} \right\}. \end{aligned}$$

Thus the equation (1.22) is a first order ordinary differential equation of  $U(u)$  as follows:

$$\begin{aligned} &\langle U(u)^{-1}U'(u)z, z \rangle \\ &= \frac{1}{2i(n+2)} \Delta G_{22} \left( E(u)^{-1}z, \overline{E(u)^{-1}z}, u \right) \\ &\quad - \frac{1}{8i(n+1)(n+2)} \langle z, z \rangle \Delta^2 G_{22} \left( E(u)^{-1}z, \overline{E(u)^{-1}z}, u \right) \\ &\quad - \frac{1}{2} \{ \langle E'(u)E(u)^{-1}z, z \rangle - \langle z, E'(u)E(u)^{-1}z \rangle \} \\ &\quad - \frac{1}{4(n+2)} \langle z, z \rangle \left\{ \text{Tr}(E'(u)E(u)^{-1}) - \overline{\text{Tr}(E'(u)E(u)^{-1})} \right\}. \end{aligned}$$

Hence by requiring

$$U(0) = E(0) = id_{n \times n},$$

there is a unique biholomorphic mapping

$$(1.23) \quad \begin{aligned} z^* &= U(w)E(w)\{z + D(z, w)\}, \\ w^* &= w, \end{aligned}$$

which transforms  $M'$  to a real hypersurface of the following form:

$$(1.24) \quad v = \langle z, z \rangle + \sum_{\min(s, t) \geq 2} H_{st}(z, \bar{z}, u)$$

where

$$\Delta H_{22} = \Delta H_{23} = 0.$$

We consider the following mappings

$$(1.25) \quad \begin{aligned} \phi_1 : & \begin{cases} z = z^* + p(w^*) \\ w = w^* + g(z^*, w^*) \end{cases} \\ \phi_2 : & \begin{cases} z^* = E(w)(z + D(z, w)) \\ w^* = w \end{cases} \\ \phi_3 : & \begin{cases} z^* = \sqrt{\text{sign}\{q'(0)\}q'(w)}Uz \\ w^* = q(w) \end{cases} \end{aligned}$$

where  $p(w)$ ,  $g(z, w)$ ,  $E(w)$ ,  $D(z, w)$ ,  $q(w)$  are holomorphic functions satisfying

$$\begin{aligned} \overline{g(0, u)} &= -g(0, u), \quad \overline{q(u)} = q(u), \\ p(0) &= q(0) = 0, \quad \det q'(0) \neq 0, \quad \det U \neq 0 \\ E(0) &= id_{n \times n}, \quad D(0, w) = D_z(0, w) = 0. \end{aligned}$$

We easily see by parameter counting that the mapping

$$(1.26) \quad (\phi_1, \phi_2, \phi_3) \longmapsto \phi_3 \circ \phi_2 \circ \phi_1$$

is bijective. Hence a biholomorphic mapping  $\phi$ ,  $\phi|_0 = 0$ , has a unique decomposition

$$\phi = \phi_3 \circ \phi_2 \circ \phi_1.$$

Note that  $U(w) = E(w) = id_{n \times n}$  and  $D(z, w) = 0$  in (1.23) whenever  $M$  is already in the form (1.24). Hence, from the decomposition (1.26), we easily see that any biholomorphic mapping, preserving the form (1.24) and the  $u$ -curve, is given by

$$(1.27) \quad \begin{aligned} z^* &= \sqrt{\text{sign}\{q'(0)\}q'(w)}Uz, \\ w^* &= q(w), \end{aligned}$$

where

$$\begin{aligned} \overline{q(w)} &= q(\bar{w}), \quad q(0) = 0, \quad q'(0) \neq 0, \\ U &\in GL(n; \mathbb{C}), \quad \langle Uz, Uz \rangle = \text{sign}\{q'(0)\}\langle z, z \rangle. \end{aligned}$$



The mapping in (1.27) transforms the real hypersurface defined by

$$\begin{aligned} v^* = & \langle z^*, z^* \rangle + H_{22}^*(z^*, \bar{z}^*, u^*) + H_{23}^*(z^*, \bar{z}^*, u^*) + H_{32}^*(z^*, \bar{z}^*, u^*) \\ & + H_{33}^*(z^*, \bar{z}^*, u^*) + O(z^{*4} \bar{z}^{*2}) + O(z^{*2} \bar{z}^{*4}) \\ & + \sum_{\min(s,t) \geq 2, s+t \geq 7} H_{st}^*(z^*, \bar{z}^*, u^*) \end{aligned}$$

to a real hypersurface as follows:

$$\begin{aligned} v = & \langle z, z \rangle + q' H_{22}^*(Uz, \overline{Uz}, q(u)) \\ & + q' \sqrt{|q'|} \{ H_{23}^*(Uz, \overline{Uz}, q(u)) + H_{32}^*(Uz, \overline{Uz}, q(u)) \} \\ & + sq'^2 H_{33}^*(Uz, \overline{Uz}, q(u)) + \left\{ \frac{1}{2} \left( \frac{q''}{q'} \right)^2 - \frac{q'''}{3q'} \right\} \langle z, z \rangle^3 \\ & + O(z^4 \bar{z}^2) + O(z^2 \bar{z}^4) \\ = & \langle z, z \rangle + H_{22}(z, \bar{z}, u) + H_{23}(z, \bar{z}, u) + H_{32}(z, \bar{z}, u) + H_{33}(z, \bar{z}, u) \\ & + O(z^4 \bar{z}^2) + O(z^2 \bar{z}^4). \end{aligned}$$

Hence we obtain

$$\begin{aligned} H_{22}(z, \bar{z}, u) &= q' H_{22}^*(Uz, \overline{Uz}, q(u)) \\ H_{23}(z, \bar{z}, u) &= q' \sqrt{|q'|} H_{23}^*(Uz, \overline{Uz}, q(u)) \\ (1.28) \quad H_{33}(z, \bar{z}, u) &= sq'^2 H_{33}^*(Uz, \overline{Uz}, q(u)) + \left\{ \frac{1}{2} \left( \frac{q''}{q'} \right)^2 - \frac{q'''}{3q'} \right\} \langle z, z \rangle^3. \end{aligned}$$

Note that  $\Delta H_{22}^* = \Delta^2 H_{23}^* = 0$  whenever  $\Delta H_{22} = \Delta^2 H_{23} = 0$ .

We can achieve the condition  $\Delta^3 H_{33}^* = 0$  by a third order ordinary differential equation as follows:

$$(1.29) \quad \frac{q'''}{3q'} - \frac{1}{2} \cdot \left( \frac{q''}{q'} \right)^2 = \kappa(u),$$

where

$$\kappa(u) = -\frac{1}{6n(n+1)(n+2)} \cdot \Delta^3 H_{33}(z, \bar{z}, u).$$

The differential equation in (1.29) determines a projective parameter on the  $u$ -curve. This completes the proof of Theorem 1.3.

**Theorem 1.4** (Chern-Moser). *Let  $M$  be a nondegenerate analytic real hypersurface defined by the equation*

$$v = F(z, \bar{z}, u) \quad F|_0 = dF|_0 = 0.$$

*Then a biholomorphic normalizing mapping of  $M$ ,  $\phi = (f, g)$  in  $\mathbb{C}^n \times \mathbb{C}$ , is uniquely determined by the value*

$$\left. \frac{\partial f}{\partial z} \right|_0, \quad \left. \frac{\partial f}{\partial w} \right|_0, \quad \Re \left( \left. \frac{\partial g}{\partial w} \right|_0 \right), \quad \Re \left( \left. \frac{\partial^2 g}{\partial w^2} \right|_0 \right).$$

As noted above, a biholomorphic mapping  $\phi$  satisfying  $\phi|_0 = 0$  is uniquely decomposed to

$$\phi = \phi_3 \circ \phi_2 \circ \phi_1$$

where  $\phi_1, \phi_2, \phi_3$  are biholomorphic mappings in (1.25) satisfying

$$\phi_1|_0 = \phi_2|_0 = \phi_3|_0 = 0.$$

Note that the mapping  $(\phi_1, \phi_2, \phi_3) \mapsto \phi = \phi_3 \circ \phi_2 \circ \phi_1$  is bijective.

We take  $\phi$  to be a normalizing biholomorphic mapping of  $M$ . Then the uniqueness of  $\phi_1, \phi_2, \phi_3$  up to the value

$$\left. \frac{\partial f}{\partial z} \right|_0, \quad \left. \frac{\partial f}{\partial w} \right|_0, \quad \Re \left( \left. \frac{\partial g}{\partial w} \right|_0 \right), \quad \Re \left( \left. \frac{\partial^2 g}{\partial w^2} \right|_0 \right)$$

assures the uniqueness of the normalizing mapping  $\phi$ . The uniqueness of  $\phi_1, \phi_2, \phi_3$  each is verified in the proof of Theorem 1.3 through uniquely determining the holomorphic functions

$$p(w), \quad E(w), \quad q(w)$$

via the ordinary differential equations (1.21), (1.22), (1.29) by the initial values

$$p'(0), \quad E(0) \equiv U, \quad q'(0), \quad q''(0).$$

We may have the following relations:

$$\begin{aligned} \sqrt{|q'(0)|}U &= \left. \frac{\partial f}{\partial z} \right|_0 \\ -\sqrt{|q'(0)|}Up'(0) &= \left(1 - i \left. \frac{\partial F}{\partial u} \right|_0\right)^{-1} \left. \frac{\partial f}{\partial w} \right|_0 \\ q'(0) &= \Re \left( \left. \frac{\partial g}{\partial w} \right|_0 \right) \\ 2q'(0)q''(0) &= \Re \left\{ \left(1 - i \left. \frac{\partial F}{\partial u} \right|_0\right)^{-2} \left. \frac{\partial^2 g}{\partial w^2} \right|_0 \right\}. \end{aligned}$$

For the case  $dF|_0 = 0$  rather than  $F_z|_0 = F_{\bar{z}}|_0 = 0$ , we have simpler relations:

$$\begin{aligned} \sqrt{|q'(0)|}U &= \left. \frac{\partial f}{\partial z} \right|_0, \quad -\sqrt{|q'(0)|}Up'(0) = \left. \frac{\partial f}{\partial w} \right|_0 \\ q'(0) &= \Re \left( \left. \frac{\partial g}{\partial w} \right|_0 \right), \quad 2q'(0)q''(0) = \Re \left( \left. \frac{\partial^2 g}{\partial w^2} \right|_0 \right) \end{aligned}$$

so that the values  $p'(0), E(0) \equiv U, q'(0), q''(0)$  are uniquely determined by

$$\left. \frac{\partial f}{\partial z} \right|_0, \quad \left. \frac{\partial f}{\partial w} \right|_0, \quad \Re \left( \left. \frac{\partial g}{\partial w} \right|_0 \right), \quad \Re \left( \left. \frac{\partial^2 g}{\partial w^2} \right|_0 \right).$$

This completes the proof of Theorem 1.4.

## 2. CHAINS AND ORBIT PARAMETERS

**I.** Let  $M$  be a nondegenerate analytic real hypersurface. Then we may define a family of distinguished curves on  $M$  via Chern-Moser normal form, which are defined alternatively and identified to be the same by E. Cartan [Ca] and Chern-Moser [CM]. Let  $\gamma : (0, 1) \rightarrow M$  be an open connected curve. Then the curve  $\gamma$  is called a chain if, for each point  $p \in \gamma$ , there exist an open neighborhood  $U$  of the point  $p$  and a biholomorphic mapping  $\phi$  on  $U$  which translates the point  $p$  to the origin and transforms  $M$  to Chern-Moser normal form such that

$$\phi(U \cap \gamma) \subset \{z = v = 0\}.$$

By Theorem 1.3 and Theorem 1.4, a chain  $\gamma$  locally exists uniquely for each vector transversal to the complex tangent plane such that  $\gamma$  is tangential to the vector.

From the proof of Theorem 1.3, we have an ordinary differential equation which locally characterizes a chain  $\gamma$ , passing through the origin  $0 \in M$ . Suppose that

$$\gamma : \begin{cases} z = p(u) \\ w = u + iF(p(u), \bar{p}(u), u) \end{cases}$$

Then there exists an ordinary differential equation

$$(2.1) \quad p'' = Q(u, p, \bar{p}, p', \bar{p}')$$

such that the function  $p(u)$  is a solution of the ordinary differential equation (2.1).

We take  $M$  to be the real hyperquadric  $v = \langle z, z \rangle$ . Then the chain  $\gamma$  is locally given by

$$\gamma : \begin{cases} z = p(u) \\ w = u + i\langle p(u), p(u) \rangle \end{cases}.$$

With  $F(z, \bar{z}, u) = \langle z, z \rangle$ , we obtain the equation  $\Delta^2 F_{23} = 0$  as follows

$$\{(1 - i\langle p', p \rangle + i\langle p, p' \rangle)p'' - ip'\langle p'', p \rangle\} + ip'\langle p, p'' \rangle = 2ip'\langle p', p' \rangle.$$

Then we easily check that the equation of chains on a real hyperquadric is given by

$$p'' = \frac{2ip'\langle p', p' \rangle (1 + 3i\langle p, p' \rangle - i\langle p', p \rangle)}{(1 + i\langle p, p' \rangle - i\langle p', p \rangle)(1 + 2i\langle p, p' \rangle - 2i\langle p', p \rangle)}.$$

**II.** The isotropy subgroup of the automorphism group of a real hyperquadric  $v = \langle z, z \rangle$  consists of fractional linear mappings  $\phi$  such that

$$(2.2) \quad \phi = \phi_\sigma : \begin{cases} z^* = \frac{C(z-aw)}{1+2i\langle z, a \rangle - w(r+i\langle a, a \rangle)} \\ w^* = \frac{\rho w}{1+2i\langle z, a \rangle - w(r+i\langle a, a \rangle)} \end{cases}$$

where the constants  $\sigma = (C, a, \rho, r)$  satisfy

$$\begin{aligned} a &\in \mathbb{C}^n, \quad \rho \neq 0, \quad \rho, r \in \mathbb{R}, \\ C &\in GL(n; \mathbb{C}), \quad \langle Cz, Cz \rangle = \rho \langle z, z \rangle. \end{aligned}$$

Further,  $\phi$  decomposes to

$$\phi = \varphi \circ \psi,$$

where

$$(2.3) \quad \psi : \begin{cases} z^* = \frac{z-aw}{1+2i\langle z, a \rangle - i\langle a, a \rangle w} \\ w^* = \frac{w}{1+2i\langle z, a \rangle - i\langle a, a \rangle w} \end{cases} \quad \text{and} \quad \varphi : \begin{cases} z^* = \frac{Cz}{1-rw} \\ w^* = \frac{\rho w}{1-rw} \end{cases}.$$

Hence the local automorphisms of a real hyperquadric is identified with a group  $H$  of the following matrices:

$$\begin{pmatrix} \rho & 0 & 0 \\ -Ca & C & 0 \\ -r - i\langle a, a \rangle & 2ia^\dagger & 1 \end{pmatrix}$$

where

$$a^\dagger = (\overline{a^1}, \dots, \overline{a^e}, -\overline{a^{e+1}}, \dots, -\overline{a^n}).$$

We easily verify

$$\phi_\sigma^*(v - \langle z, z \rangle) = (v - \langle z, z \rangle)\rho(1 + \delta)^{-1}(1 + \bar{\delta})^{-1},$$

where

$$1 + \delta = 1 + 2i\langle z, a \rangle - (r + i\langle a, a \rangle)w.$$

Hence the automorphisms  $\phi_\sigma$  are normalizations of a real hyperquadric. Further, by Theorem 1.4, each normalization of a real hyperquadric is necessarily an automorphism. Then a chain  $\gamma$  on a real hyperquadric is necessarily given by

$$\begin{aligned} \gamma &= \phi^{-1}(z = v = 0) \\ &= \left\{ \left( \frac{\rho^{-1}ua}{1-\rho^{-1}u(-r+i\langle a, a \rangle)}, \frac{\rho^{-1}u}{1-\rho^{-1}u(-r+i\langle a, a \rangle)} \right) \right\} \\ &= \{v = \langle z, z \rangle\} \cap \mathbb{C}(a, 1) \end{aligned}$$

so that the chain  $\gamma$  is just an intersection of a complex line.

By Theorem 1.4, each normalization  $N = (f, g)$  is uniquely determined by the initial value

$$C, \quad a, \quad \rho, \quad r$$

such that

$$\begin{aligned} f(z, w) &= C(z - aw) + f^*(z, w) \\ g(z, w) &= \rho(w + rw^2) + g^*(z, w) \end{aligned}$$

where

$$f^*|_0 = df^*|_0 = g^*|_0 = dg^*|_0 = \Re(g_{ww}^*|_0) = 0.$$

Hence the group  $H = \{(C, a, \rho, r)\}$  parameterizes the normalizations of a real hypersurface. Further, Theorem 1.3 and Theorem 1.4 together yields a family of polynomial identities(cf. [Pa1]). Then we have showed that the group  $H$  gives a group action via normalization on the class of normalized real hypersurfaces.

**III.** Suppose that  $M$  is an analytic real hypersurface defined near the origin by the following equation:

$$v = \langle z, z \rangle + \sum_{\alpha, \beta} \left( \kappa_{\alpha\beta} z^\alpha z^\beta + \kappa_{\bar{\alpha}\bar{\beta}} \bar{z}^\alpha \bar{z}^\beta \right) + F(z, \bar{z}, u)$$

where

$$F(z, \bar{z}, u) = \sum_{k=3}^{\infty} F_k(z, \bar{z}, u).$$

Let  $N_\sigma$  be a normalization of  $M$  with the initial value  $\sigma = (C, a, \rho, r) \in H$  and let  $\phi_{\sigma'} = \varphi \circ \psi$  be a local automorphism of a real hyperquadric(cf. (2.2), (2.3)) with the initial value  $\sigma' = (C, a, \rho, r_0) \in H$ , where

$$r_0 = r - \Re(\kappa_{\alpha\beta} a^\alpha a^\beta).$$

Then there are two decompositions of  $N_\sigma$  as follows(cf. [CM]):

$$\begin{cases} N_\sigma = E \circ \phi_{\sigma'} = E \circ \varphi \circ \psi, \\ N_\sigma = \varphi \circ E \circ \psi, \end{cases}$$

where  $E$  is the normalization with the identity initial value.

Let  $M$  be a nondegenerate real hypersurface and  $N_\sigma$  be a normalization of  $M$  with initial value  $\sigma = (C, a, \rho, r) \in H$  such that  $M' \equiv N_\sigma(M)$  is defined by the equation

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}^*(z, \bar{z}, u)$$

where

$$\Delta F_{22}^* = \Delta^2 F_{23}^* = \Delta^3 F_{33}^* = 0.$$

Note that the mapping  $\varphi$  is itself a normalization in the following decomposition:

$$(2.4) \quad \begin{aligned} N_\sigma &= \varphi \circ E \circ \psi \\ &= \varphi \circ N_{\sigma_1} \end{aligned}$$

where  $N_{\sigma_1}$  is a normalization of  $M$  with initial value  $\sigma_1 = (id_{n \times n}, a, 1, 0) \in H$ .

As a consequences of the decomposition (2.4), we notice that a normalization  $N_\sigma$  is analytic of

$$z, \quad w, \quad C, \quad \rho, \quad r$$

near the point  $z = w = r = 0$  and  $C = id_{n \times n}, \rho = 1$ . More precisely,

$$N_\sigma = \left( \frac{Cf(z, w)}{1 - rg(z, w)}, \frac{\rho g(z, w)}{1 - rg(z, w)} \right)$$

where

$$N_{\sigma_1} = (f(z, w), g(z, w)) \quad \sigma_1 = (id_{n \times n}, a, 1, 0).$$

Notice that the size of convergence of the normalization  $N_\sigma$  at the origin is determined by the value  $a, r$ .

Further, suppose that the transformed real hypersurface  $N_\sigma(M)$  is defined by

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u).$$

Then the function  $F^*(z, \bar{z}, u)$  is real-analytic of

$$z, \quad w, \quad C, \quad \rho, \quad r$$

near the point  $z = w = r = 0$  and  $C = id_{n \times n}, \rho = 1$ .

In fact, from the proof of Theorem 1.3, we obtain

**Theorem 2.1.** *Let  $M$  be a nondegenerate real-analytic real hypersurface defined by*

$$v = F(z, \bar{z}, u) \quad F|_0 = F_z|_0 = F_{\bar{z}}|_0 = 0.$$

*Then  $N_\sigma$  and  $F^*(z, \bar{z}, u)$  are analytic of*

$$z, \quad w, \quad C, \quad a, \quad \rho, \quad r.$$

We have a natural group action by normalizations on the class of real hypersurfaces in normal form(cf. [Pa1]). Then, under a natural compact-open topology(cf. [Na]), we obtain

**Corollary 2.2.** *The local automorphism group of a nondegenerate analytic real hypersurface is a Lie group.*

## 3. NORMAL FORMS OF REAL HYPERSURFACES

I. Let  $M$  be a nondegenerate analytic real hypersurface defined by the equation

$$v = F_{11}^*(z, \bar{z}, u) + \sum_{s,t \geq 2} F_{st}^*(z, \bar{z}, u).$$

Let's take a matrix valued function  $E(u)$  satisfying

$$F_{11}^*(z, \bar{z}, u) = \langle E(u)z, E(u)z \rangle.$$

Then suppose that the biholomorphic mapping

$$\phi : \begin{cases} z^* = E(u)z \\ w^* = w \end{cases}$$

transforms  $M$  to another real hypersurface  $\phi(M)$  defined by

$$v = F_{11}^*(z, \bar{z}, u) + \sum_{s,t \geq 2} G_{st}(z, \bar{z}, u).$$

Then we have the following relation

$$\begin{aligned} & F_{11}^*(E(u)^{-1}E'(u)z, \bar{z}, u) \\ = & -\frac{2i}{n+2} \cdot \text{tr} G_{22}(z, \bar{z}, u) + \frac{i}{(n+1)(n+2)} \cdot (\text{tr})^2 G_{22}(z, \bar{z}, u) \cdot F_{11}^*(z, \bar{z}, u) \\ & + \frac{2i}{n+2} \cdot \text{tr} F_{22}^* \left( E_1(u)^{-1}E(u)z, \overline{E_1(u)^{-1}E(u)z}, u \right) \\ & - \frac{i}{(n+1)(n+2)} \cdot (\text{tr})^2 F_{22}^*(z, \bar{z}, u) \cdot F_{11}^*(z, \bar{z}, u) \\ & + \frac{1}{2} \left( \frac{\partial F_{11}^*}{\partial u} \right) (z, \bar{z}, u). \end{aligned}$$

where  $E_1(u)$  is a given matrix valued function satisfying

$$F_{11}^*(z, \bar{z}, u) = \langle E_1(u)z, E_1(u)z \rangle.$$

Here we have constant solutions

$$E(u) = E(0)$$

whenever

$$\text{tr} F_{22}^*(z, \bar{z}, u) = \text{const.} F_{11}^*(z, \bar{z}, u) = \text{tr} G_{22}(z, \bar{z}, u)$$

and

$$\left( \frac{\partial F_{11}^*}{\partial u} \right) (z, \bar{z}, u) = 0.$$

As a necessary condition for a normal form, we require that the  $u$ -curve be a chain. From the observation in the previous paragraph, a normal form may have to take the following form

$$v = \langle z, z \rangle + \sum_{s,t \geq 2} F_{st}^*(z, \bar{z}, u)$$

where

$$\begin{aligned} \Delta F_{22}^*(z, \bar{z}, u) &= \text{const.} \langle z, z \rangle \\ \Delta^2 F_{23}^*(z, \bar{z}, u) &= 0. \end{aligned}$$

Suppose that a real hypersurface  $M$  is defined by the equation

$$(3.1) \quad \begin{aligned} v &= \langle z, z \rangle + \sum_{s,t \geq 2} G_{st}(z, \bar{z}, u) && \text{for } \alpha = 0 \\ v &= -\frac{1}{2\alpha} \ln \{1 - 2\alpha \langle z, z \rangle\} + \sum_{s,t \geq 2} G_{st}(z, \bar{z}, u) && \text{for } \alpha \neq 0 \end{aligned}$$

where

$$\Delta G_{22}(z, \bar{z}, u) = \Delta^2 G_{23}(z, \bar{z}, u) = 0.$$

Let  $\varphi$  be a biholomorphic mapping leaving the  $u$ -curve invariant and preserving the form (3.1). Then  $\varphi$  is necessarily given by the following mapping(cf. [Pa2]):

$$\varphi : \begin{cases} z^* = \sqrt{\text{sign}\{q'(0)\} q'(w)} U z \exp \frac{\alpha i}{2} (q(w) - w) \\ w^* = q(w) \end{cases}$$

where  $\alpha \in \mathbb{R}$  and

$$\langle U z, U z \rangle = \text{sign}\{q'(0)\} \langle z, z \rangle.$$

Suppose that the biholomorphic mapping  $\varphi$  transforms  $M$  to a real hypersurface  $\varphi(M)$  defined by

$$v = -\frac{1}{2\alpha} \ln \{1 - 2\alpha \langle z, z \rangle\} + \sum_{s,t \geq 2} G_{st}^*(z, \bar{z}, u)$$

where

$$\Delta G_{22}^*(z, \bar{z}, u) = \Delta^2 G_{23}^*(z, \bar{z}, u) = 0.$$

Then we have the following relation

$$\begin{aligned} q'(u) G_{22}^*(U z, \overline{U z}, q(u)) &= G_{22}(z, \bar{z}, u) \\ q'(u) \sqrt{|q'(u)|} \exp -\frac{\alpha i}{2} (q(u) - u) G_{23}^*(U z, \overline{U z}, q(u)) &= G_{23}(z, \bar{z}, u) \end{aligned}$$

and

$$\begin{aligned} &\frac{q'''(u)}{3q'(u)} - \frac{1}{2} \left( \frac{q''(u)}{q'(u)} \right)^2 + \frac{\alpha^2}{6} (q'(u)^2 - 1) \\ &= \frac{1}{6n(n+1)(n+2)} \{ q'(u)^2 \Delta^3 G_{33}^*(z, \bar{z}, q(u)) - \Delta^3 G_{33}(z, \bar{z}, u) \}. \end{aligned}$$

Notice that

$$(3.2) \quad \frac{q'''}{3q'} - \frac{1}{2} \left( \frac{q''}{q'} \right)^2 + \frac{\alpha^2}{6} (q'^2 - 1) = 0$$

whenever

$$\Delta^3 G_{33}^*(z, \bar{z}, q) dq^2 = \Delta^3 G_{33}(z, \bar{z}, u) du^2.$$

We want to restrict the mapping  $\varphi$  so that the function  $q(u)$  is a solution of the ordinary differential equation (3.2). The restriction on  $\varphi$  has to be achieved by requiring an additional condition on the normal form (3.1).

We claim that the following choice works

$$(3.3) \quad \begin{aligned} \Delta^3 G_{33}(z, \bar{z}, u) &= \text{const.} \Delta^4 (G_{22}(z, \bar{z}, u))^2 \\ \Delta^3 G_{33}^*(z, \bar{z}, q) &= \text{const.} \Delta^4 (G_{22}^*(z, \bar{z}, q))^2. \end{aligned}$$

Because of the relation

$$G_{22}^*(z, \bar{z}, q) dq = G_{22}(z, \bar{z}, u) du,$$

the condition (3.3) gives

$$q'(u)^2 \Delta^3 G_{33}^*(z, \bar{z}, q(u)) = \Delta^3 G_{33}(z, \bar{z}, u)$$

which yields the ordinary differential equation (3.2).

Hence we define a normal form such that

$$\begin{aligned} v &= \langle z, z \rangle + \sum_{s,t \geq 2} G_{st}(z, \bar{z}, u) & \text{for } \alpha = 0 \\ v &= -\frac{1}{2\alpha} \ln \{1 - 2\alpha \langle z, z \rangle\} + \sum_{s,t \geq 2} G_{st}(z, \bar{z}, u) & \text{for } \alpha \neq 0 \end{aligned}$$

where

$$\begin{aligned} \Delta G_{22}(z, \bar{z}, u) &= \Delta^2 G_{23}(z, \bar{z}, u) = 0 \\ \Delta^3 G_{33}(z, \bar{z}, u) &= \beta \Delta^4 (G_{22}(z, \bar{z}, u))^2 \quad \text{for some } \beta \in \mathbb{R}. \end{aligned}$$

We easily see that all normalizations associated to any normal form above are uniquely determined by some constant initial values.

Chern-Moser normal form is given in the case of  $\alpha = \beta = 0$  so that

$$v = \langle z, z \rangle + \sum_{s,t \geq 2} G_{st}(z, \bar{z}, u)$$

where

$$\Delta G_{22} = \Delta^2 G_{23} = \Delta^3 G_{33} = 0.$$

Moser-Vitushkin normal form is defined by taking  $\alpha \neq 0$  and  $\beta = 0$  so that

$$v = -\frac{1}{2\alpha} \ln \{1 - 2\alpha \langle z, z \rangle\} + \sum_{s,t \geq 2} G_{st}(z, \bar{z}, u)$$

where

$$\Delta G_{22} = \Delta^2 G_{23} = \Delta^3 G_{33} = 0.$$

We shall see each normal form has its own advantage in applications(cf. [Pa2]).

**II.** Burns and Shnider [BS] have reported that the geometric theory of Chern and Moser [CM] gives a projective parametrization on a chain which is different from the parametrization defined by Chern-Moser normal form(cf. [BFG]). From (1.28), we obtain

$$\begin{aligned} & \frac{q'''}{3q'} - \frac{1}{2} \cdot \left( \frac{q''}{q'} \right)^2 \\ (3.4) \quad &= \frac{1}{6n(n+1)(n+2)} \{q'^2 \Delta^3 H_{33}^*(z, \bar{z}, q(u)) - \Delta^3 H_{33}(z, \bar{z}, u)\}. \end{aligned}$$

Note that the solution  $q(u)$  in (3.4) is given by

$$q(u) = \frac{au + b}{cu + d}, \quad ad - bc \neq 0, \quad a, b, c, d \in \mathbb{R}$$

whenever

$$(3.5) \quad \varepsilon \equiv \Delta^3 H_{33}(z, \bar{z}, u) du^2 = \Delta^3 H_{33}^*(z, \bar{z}, q) dq^2.$$

The mapping (1.27) effects

$$H_{2k}(z, \bar{z}, u) = q' |q'|^{\frac{k-2}{2}} H_{2k}^*(Uz, \overline{Uz}, q(u)) \quad \text{for } k \geq 2.$$



Thus there are many possible choices satisfying the equalities (3.5). To reduce the possible choices, we require that the function  $\varepsilon$  is independent of a choice of a chain. Clearly there exists such a function  $\varepsilon$ , for instance,  $\varepsilon = 0$  so that

$$\Delta^3 H_{33}^*(z, \bar{z}, q(u)) = \Delta^3 H_{33}(z, \bar{z}, u) = 0.$$

The requirement is also satisfied by a function  $\varepsilon$  if we define  $\varepsilon$  as follows:

$$\begin{aligned}\varepsilon &\equiv \Delta^3 H_{33} du^2 \\ \Delta^3 H_{33} &= c \Delta^4 (H_{22})^2,\end{aligned}$$

for a constant real number  $c \in \mathbb{R}$ . Thus we can define a normal form similar to Chern-Moser normal form except for replacing the condition  $\Delta^3 H_{33} = 0$  with

$$(3.6) \quad \sum N_{\alpha\beta\gamma\dots}^{\alpha\beta\gamma} = \frac{4}{9} \sum N_{\alpha\beta\dots}^{\gamma\delta} N_{\gamma\delta\dots}^{\alpha\beta}$$

where

$$\begin{aligned}H_{22}(z, \bar{z}, u) &= \sum N_{\alpha\beta\bar{\gamma}\bar{\delta}} z^\alpha z^\beta \bar{z}^\gamma \bar{z}^\delta \\ H_{23}(z, \bar{z}, u) &= \sum N_{\alpha\beta\gamma\bar{\delta}\bar{\rho}\bar{\sigma}} z^\alpha z^\beta z^\gamma \bar{z}^\delta \bar{z}^\rho \bar{z}^\sigma \\ \sum N_{\alpha\beta\gamma\dots}^{\alpha\beta\gamma} &= \frac{1}{(3 \cdot 2)^2} \Delta^3 H_{33} \\ \sum N_{\alpha\beta\dots}^{\gamma\delta} N_{\gamma\delta\dots}^{\alpha\beta} &= \frac{1}{3 \cdot 2^5} \Delta^4 (H_{22})^2.\end{aligned}$$

Then the condition (3.6) gives on a chain the parametrization of the geometric theory of Chern and Moser(cf. [Fa]).

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